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Theory of Complex Variables - MA 209
Problem Sheet - 4

Powers and Roots

1. For the following problems compute all roots. Give the principal nth root in each case. Sketch the roots $w_0, w_1, ..., w_{n-1}$ on an appropriate circle centered at the origin.

   (a) $(8)^{\frac{1}{3}}$

   (b) $(-125)^{\frac{1}{3}}$

   (c) $(-1+i)^{\frac{1}{3}}$

   (d) $(-1-\sqrt{3}i)^{\frac{1}{4}}$

   (e) $(\frac{1+i}{\sqrt{3}+i})^{(1/6)}$

2. Use the fact that $8i = (2+2i)^2$ to find all solutions of the equation $z^2 - 8z + 16 = 8i$.

3. Show that the n nth roots of unity are given by
   $(1)^{(1/n)} = \cos(\frac{2k\pi}{n}) + i\sin(\frac{2k\pi}{n})\ k = 0, 1, 2, ..., n-1$
   (a)Find n nth roots of unity for $n = 3, n = 4, n = 5$
   (b)Carefully plot the roots of unity found in part (b).Sketch the regular polygons formed with the roots as vertices.

4. Suppose ω is a cube root of unity corresponding to k = 1 in the last problem.
   (a) How are ω and ω^2 related?
   (b)Verify by direct computation that $1 + \omega + \omega^2 = 0$.
   (c) Explain how the result in part (b) follows from the basic definition that ω is a cube root of 1, that is, $\omega^3 = 1$. [Hint: Factor]

5. For a fixed n, if we take k = 1 in Problem 22 , we obtain the root $\omega_n = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$
   Explain why the n nth roots of unity can then be written $1, \omega_n, \omega_n^2, ..., \omega_n^{n-1}$

6. Consider the equation $(z+2)^n + z^n = 0$, where n is a positive integer. By any means, solve the equation for z when n = 1 and n = 2.

7. Consider the equation in Problem 25.
   (a) In the complex plane, determine the location of all solutions z when n = 5. [Hint: Write the equation in the form $[(z+2)/(-z)]^5 = 1$ and use part (a) of Problem 22.]
   (b) Reexamine the solutions of the equation in Problem 25 for n = 1 and n = 2.

8. Let n be a fixed natural number.Put $\omega_n = cis(\frac{2\pi}{n})$. Show that $1 + \omega_n + \omega_n^2 + \omega_n^3 + \ + \omega_n^{n-1} = 0$. [Hint: Multiply the sum $1 + \omega_n + \omega_n^2 + \omega_n^3 + \ + \omega_n^{n-1}$ by $\omega_n - 1$.]

9. Suppose n denotes a nonnegative integer. Determine the values of n such that $z^n = 1$ possesses only real solutions. Defend your answer with sound mathematics.

10. Discuss: A real number can have a complex nth root. Can a nonreal complex number have a real nth root?

11. Suppose w is located in the first quadrant and is a cube root of a complex number z. Can there exist a second cube root of z located in the first quadrant? Defend your answer with sound mathematics.

12. Suppose z is a complex number that possesses a fourth root w that is neither real nor pure imaginary. Explain why the remaining fourth roots are neither real nor pure imaginary.

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